can now prove an interesting proposition. While we will soon reduce this statement to an immediate corollary of a more powerful statement, the following proof is quite illustrative of the general techniques we will use in this section.

Proposition. *All positive integers are interesting.*

Proof. In the case $n = 1$, n is the unique number with the property that $x \cdot n = n \cdot x = x$ for all $x \in \mathbf{R}$. This is an interesting property, so 1 is interesting. Now assume that the integers $1, 2, \ldots, n-1$ are all interesting. If n were not interesting, it would be the least non-interesting number. But this is itself an interesting property, so n must be interesting.

For many years, mathematicians have struggled with a now famous conjecture, which stated simply that "everyone sleeps with Ben." The conjecture remains undecided, but mathematical induction has been used to gain valuable insight into the problem.

Lemma 1. *If Ben sleeps, then everyone sleeps with Ben.*

Proof. Let $P(n)$ be the proposition that given any group of n people which included Ben, all members of the group sleep with Ben. By hypothesis Ben sleeps, hence Ben sleeps with Ben, so $P(1)$ holds. Now suppose $P(n-1)$ holds and consider a group of n people including Ben. Every member of the group also belongs to a subgroup of $n-1$ people (delete an arbitrary non-Ben person), hence every member of the group sleeps with Ben. Therefore $P(n)$ holds, so by induction everyone sleeps with Ben.

Lemma 2. *If everyone sleeps with Ben, then Ben does not sleep.*

Proof. There are more than five billion people in the world. Nobody can sleep in a bed with more than 4,999,999,999 people, so Ben doesn't sleep.

Thus, the Ben sleeps only if Ben sleeps with everyone only if Ben does not sleep, a contradiction. We have just proved:

Theorem 1.1 (Sleeplessness). *Ben does not sleep.*

§3. More on Induction

Proposition. *All positive integers are equal.*

Proof. It suffices to show that for all $n \in \mathbf{N}$, if a and $b \in \mathbf{N}$ satisfy $\max(a, b) = n$ then $a = b$. We proceed by induction.

In the case $n = 1$, then $a = b = 1$. Now, assume that the result holds for $n = k$ and take a and b sucj that $\max(a, b) = k + 1$. Then $\max(a - 1, b - 1) = k$ so by hypothesis $a - 1 = b - 1$ and hence $a = b$.